

Gaps in Algebraic and Differentially Linear Series

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Abstract

It is proved that coefficients of algebraic or differentially linear power series satisfy linear difference equations with polynomial coefficients. This is used to exhibit a bound for the gap among the coefficients of power series which are algebraic or differentially linear. The bound obtained is absolute and depends only on the power series and its defining differential equation.

บทคัดย่อ

งานนี้เป็นการพิสูจน์ว่า สัมประสิทธิ์ของอนุกรมกำลังพีชคณิต หรืออนุกรมกำลังเชิงอนุพันธ์เชิงเส้น สอดคล้องกับสมการผลต่างเชิงเส้นที่มีสัมประสิทธิ์เป็นพหุนาม จากนั้น เป็นการนำผลดังกล่าวนี้ในการหาขอบเขตสำหรับช่องว่างระหว่างสัมประสิทธิ์ในอนุกรมกำลังที่เป็นอนุกรมพีชคณิต หรือเป็นอนุกรมเชิงอนุพันธ์เชิงเส้น ขอบเขตที่ได้รับเป็นแบบสัมบูรณ์ และขึ้นเพียงกับอนุกรมกำลังและสมการเชิงอนุพันธ์ที่นิยามอนุกรมนั้น

Introduction

It has long been known that formal power series with large gaps among their coefficients are transcendental (see e.g. Mahler (1976)). Indeed, power series with very large gaps besides being transcendental are not even differentially algebraic (see Ostrowski (1920), Popken (1935), Mahler (1976), Sibuya and Sperber (1981), Lipshitz and Rubel (1986)). Here by differentially algebraic series, we mean formal power series which satisfy nontrivial algebraic differential equations. Consequently, there arises the problem of determining how large the gaps ought to be in order that a power series should be of certain particular kind. To date, this problem has not fully been settled. Some of the most interesting results are as follows:

Let

$$y = \sum_{k=0}^{\infty} f(n_k)z^{n_k}, \quad f(n_k) \neq 0$$

be a gap series. Define

$$\Delta_k = n_{k+1} - n_k,$$

and let $d(k)$ be the largest integer less than or equal to k such that

$$\Delta_k \geq n_{d(k)}$$

Maillet (1903), Ostrowski (1920) and Popken (1935) showed that if

$$\limsup_k \frac{n_{k+1}}{n_k} < \infty, \text{ then } y \text{ is not differentially algebraic.}$$

The sharpest result in this direction so far is due to Lipshitz and Rubel (1986) who proved that if y is differentially algebraic, then

$$\lim_{k \rightarrow \infty} \frac{d(k)}{k} \neq 1.$$

In this paper, we refine the work of Lipshitz and Rubel slightly by examining the cases of algebraic and differentially linear series. By differentially linear series, we mean formal power series which satisfy linear differential equations with polynomial coefficients. For convenience, we take the complex number field \mathbb{C} as our underlying field; though, the entire discussion holds in any field of characteristic 0. Our principal result reads :

Theorem. Let $y = \sum_{k=0}^{\infty} f(n_k)z^{n_k}$, $f(n_k) \neq 0$ be a formal gap series.

Define $\Delta_k = n_{k+1} - n_k$, and define $d(k)$ to be the largest integer less than or equal to k such that $\Delta_k \geq n_{d(k)}$.

If y is algebraic or is differentially linear, then Δ_k and $d(k)$ are bounded, and so $\lim_{k \rightarrow \infty} d(k)/k = 0$.

Materials and Methods

The proof of the principal theorem is based on the following auxiliary lemmas. The former is classical and can be found in Mahler (1976) or Stanley (1980), While the latter is partly due to Popken (1935). For the sake of completeness, we make available here all relevant proofs.

Lemma 1. If y is an algebraic power series of degree $d(\geq 2)$ over $\mathbb{C}(z)$, then y satisfies a linear differential equation of order at most $d-1$ and with coefficients in $\mathbb{C}(z)$.

Proof. (Mahler (1976)). Let

$$(1) \quad F(y) := y^d + a_{d-1}(z)y^{d-1} + \dots + a_1(z)y + a_0(z) = 0$$

be the minimal polynomial for y . Thus $a_i(z)$, $i = 0, 1, \dots, d$, are elements of $\mathbb{C}(z)$, and $F(y)$ is irreducible over $\mathbb{C}(z)$. Differentiating (1) with respect to z , we get.

$$(2) \quad F_y(y) y' + G_0(y) = 0,$$

where $F_y := \partial F / \partial y$ and G_0 is a polynomial in y with coefficients in $\mathbb{C}(z)$. Since F_y is not identically zero and F is irreducible, then F and F_y are relatively prime, and the Euclid algorithm shows that there exist two nonzero elements of $\mathbb{C}(z)[y]$ say $a(y)$ and $b(y)$, such that

$$a(y) F(y) + b(y) F_y(y) = 1.$$

Using (1), we see that $F_y(y)$ has a polynomial in y (with coefficients in $\mathbb{C}(z)$) as its reciprocal. Multiplying (2) by $b(y)$, we obtain an equation

$$(3) \quad y' = G_1(y),$$

where G_1 is a polynomial in y with coefficients in $C(z)$. We can assume G_1 to be of degree at most $d-1$ since we can reduce it (mod F). Differentiating (3) with respect to z and using (1), (2), we get

$$y'' = G_2(y),$$

where G_2 is again a polynomial in y with coefficients in $C(z)$ of degree at most $d-1$. Proceeding in this way, we obtain a system

$$(4) \quad y^{(k)} = G_k(y) \quad (k = 1, 2, \dots, d-1),$$

where the G_k 's are polynomials in y of degree at most $d-1$ and with coefficients in $C(z)$. The system (4) can be written as

$$(5) \quad \begin{cases} R_{11}y + R_{12}y^2 + \dots + R_{1,d-1}y^{d-1} = y' - R_1 \\ R_{21}y + R_{22}y^2 + \dots + R_{2,d-1}y^{d-1} = y'' - R_2 \\ \vdots \\ R_{d-1,1}y + R_{d-1,2}y^2 + \dots + R_{d-1,d-1}y^{d-1} = y^{(d-1)} - R_{d-1} \end{cases}$$

Where the R_{ij} and R_i ($1 \leq i, j \leq d-1$) are elements of $C(z)$.

We now distinguish two cases

Case (i) $\det R_{ij}$ is not identically zero. Then we can solve the system (5) for y, y^2, \dots, y^{d-1} in terms of the right hand sides. In particular, we have

$$y = b_0 + b_1y' + b_2y'' + \dots + b_{d-1}y^{(d-1)},$$

where the b_i 's are elements of $C(z)$, and this is a linear differential equation sought after.

Case (ii) $\det R_{ij}$ is identically zero. Then the right hand sides of (5) are linearly dependent over $C(z)$. Thus there are rational functions b_1, \dots, b_{d-1} in $C(z)$, not all identically zero, such that

$$b_1(y' - R_1) + b_2(y'' - R_2) + \dots + b_{d-1}(y^{(d-1)} - R_{d-1}) = 0,$$

and again this is a linear differential equation sought after.

Lemma 2. Let $y = \sum_{k=0}^{\infty} f(k)z^k$ be a power series with complex coefficients.

Assume that y satisfies a linear differential equation of the form

$$(6) \quad p_m(z)y^{(m)} + p_{m-1}(z)y^{(m-1)} + \dots + p_1(z)y' + p_0(z)y = 0$$

where m is a positive integer, the p_i 's are elements of $C[z]$ with $\deg p_i(z) \leq g$ for all i , and $p_m(z)$ not being identically zero. Then there exists a nonnegative integer $K (\leq g)$ such that the coefficients $f(n)$, $n \geq 0$, satisfy a linear difference equation of degree $m + g - K$ of the form

$$a_0(n)f(n) + a_1(n)f(n+1) + \dots + a_{m+g-K}(n)f(n+m+g-K) = 0,$$

where the a_i 's are polynomials in n , depending on p_0, \dots, p_m , of degree at most $g+i$ with complex coefficients and not all of them are identically vanishing.

Proof. Writing D for d/dz , for each nonnegative integer $j \geq g$, we have

$$D^j(p_i(z)y^{(i)}(z)) = \sum_{k=0}^g \binom{j}{k} p_i^{(k)}(z) y^{(i+j-k)}(z) \quad (i = 0, 1, \dots, m).$$

Taking the j th derivative of the expression in (6), and using this last identity, we get

$$0 = \sum_{k=0}^{m+g} \left(\sum_{s=0}^g \binom{j}{s} p_{m-k+s}^{(s)}(z) \right) y^{(m+j-k)}(z).$$

Since $p_m(z)$ is not identically zero, let $M (\leq g)$ be the least nonnegative integer for which

$$p_m^{(M)} \neq 0.$$

Putting $z = 0$, we get for nonnegative integer j

$$(7) \quad \sum_{k=0}^{m+g} F_k(j) y^{(m+j-k)}(0) = 0,$$

$$\text{where } F_k(j) = \sum_{s=0}^g \binom{j}{s} p_{m-k+s}^{(s)}(0).$$

Observe that $F_k(j)$ is a polynomial in j of degree at most g . Since the leading coefficient of $F_m(j)$ is $p_m^{(M)} / m! \neq 0$, then not all $F_k(j)$ are identically zero. Let K , $0 \leq K \leq M$, be the least nonnegative integer, independent of j , such that $F_K(j)$ is not identically zero as a polynomial in j . Thus for $j \geq g$, the equation (7) yields

$$\sum_{k=K}^{m+g} F_k(j) (m+j-k)! f(m+j-k) = 0.$$

Dividing by $(j-g)!$, we get

$$\sum_{k=0}^{m+g-K} b_{m+g-k}(j) f(j-g+k) = 0.$$

where $b_{m+g-k}(j) = F_{m+g-k}(j) (j-g+k)! / (j-g)!$ is a polynomial in j of degree at most $g+k$ with complex coefficients, and not all $b_{m+g-k}(j)$ are identically zero. Putting

$$n = j-g \geq 0, \quad a_k(n) = b_{m+g-k}(j),$$

we obtain the desired difference equation of the form

$$\sum_{k=0}^{m+g-K} a_k(n) f(n+k) = 0,$$

where the $a_k(n)$'s are polynomials in n of degree at most $g+k$ and are dependent on $p_0(z), \dots, p_m(z)$.

Results

Proof of the theorem. Either y is algebraic or y is differentially linear, by Lemma 1, we can assume that y satisfies a nontrivial linear differential equation (6) of exact order m with coefficients in $C[z]$ each of degree not exceeding g . Next, by Lemma 2, we see that the coefficients $f(n_k)$ of the series y satisfy a nontrivial linear difference equation of order $m+g-K$ with polynomial (in n) coefficients, where K is a fixed nonnegative integer not greater than g .

Since y is a nonzero gap series, it follows from this difference equation that the gap Δ_k ($:= n_{k+1} - n_k$) among the coefficients $f(n_k)$ of y must satisfy

$$\Delta_k \leq m+g-K \quad (k = 0, 1, 2, \dots).$$

Using the definition of $d(k)$, this then implies that $d(k)$ ($\leq n_{d(k)} \leq \Delta_k$) is bounded, for each k , by $m+g-K$ and consequently

$$\lim_{k \rightarrow \infty} d(k)/k = 0,$$

as required.

Discussion

For the case where y is an algebraic series, there is another proof of the results in Lemmas 1 and 2, which is somewhat conceptually simpler but contains less information. This proof is due to Stanley (1980) and we give a sketch here for comparison purposes.

Let y satisfy the defining equation (1) above. By differentiating (1) repeatedly with respect to z and using induction, we get that $y^{(k)}$ is a rational function $R_k(z, y)$ of z and y with complex coefficients for all nonnegative integer k . Since y is algebraic of degree d over $C(z)$, the functions $1, y, y', \dots, y^{(d-1)}$ are linearly dependent over $C(z)$. Writing this dependence relation gives the result of Lemma 1. Now clear the denominators of the dependence relation so the coefficients of each $y^{(k)}$ is an element of $C[z]$, and expand each $y^{(k)}$ as a power series in z . Equate the coefficients of z^n on both sides of the dependence relation to get the desired result of Lemma 2.

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